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Tau-functions and Dressing Transformations for Zero-Curvature Affine Integrable Equations

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ABSTRACT

The solutions of a large class of hierarchies of zero-curvature equations that includes Toda and KdV type hierarchies are investigated. All these hierarchies are constructed from affine (twisted or untwisted) Kac-Moody algebras $\mathfrak{g}$. Their common feature is that they have some special "vacuum solutions" corresponding to Lax operators lying in some abelian (up to the central term) subalgebra of $\mathfrak{g}$; in some interesting cases such subalgebras are of the Heisenberg type. Using the dressing transformation method, the solutions in the orbit of those vacuum solutions are constructed in a uniform way. Then, the generalized tau-functions for those hierarchies are defined as an alternative set of variables corresponding to certain matrix elements evaluated in the integrable highest-weight representations of $\mathfrak{g}$. Such definition of tau-functions applies for any level of the representation, and it is independent of its realization (vertex operator or not). The particular important cases of generalized mKdV and KdV hierarchies as well as the abelian and non abelian affine Toda theories are discussed in detail.

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1. Introduction.

In this paper we shall be concerned with the generalization of the Hirota method of constructing the solutions of hierarchies of non-linear integrable models. In particular, we shall study such connections through a large and important class of solutions which can be constructed in a uniform way using the underlying structure of affine Kac-Moody algebras of those hierarchies.

Out of the different available methods for solving integrable partial differential equations, the Hirota method has proved to be particularly useful. This method started with the work of R. Hirota [1], who discovered a way to construct various types of explicit solutions to the equations, and, in particular, their multiple soliton solutions. The idea is to find a new set of variables, called the "tau-functions", which then satisfy simpler — originally bilinear— equations known as Hirota equations. For instance, the tau-function of the Korteweg-de Vries equation (KdV), $\partial_t u = \partial_x^3 u + 6u\, \partial_x u$, is related to the original variable by the celebrated formula

$$u = 2\, \partial_x^2 \ln \tau \,, \tag{1.1}$$

Such tau-function satisfies a bilinear Hirota equation [2], and the exact multi-soliton solutions are found by considering truncated series expansions of τ in some arbitrary parameter ϵ, *e.g.*, $\tau = 1 + \epsilon\, \tau^{(1)} + \cdots \epsilon^n\, \tau^{(n)}$.

More recently, the Hirota method has been used to obtain the multi-soliton solutions of affine (abelian) Toda equations [3,4,5,6] (the method was originally applied to the sine-Gordon equation in ref. [7]). The success of the method depends crucially on the choice of the change of variables between the Toda fields $\boldsymbol{\phi}$ and the Hirota's tau functions τ_i, namely

$$\boldsymbol{\phi} = -\sum_{i=0}^{r} \frac{2}{\boldsymbol{\alpha}_i^2}\, \boldsymbol{\alpha}_i\, \ln \tau_i \,, \tag{1.2}$$

where $\boldsymbol{\alpha}_i$ are the simple roots of the associated affine untwisted Kac-Moody algebra.

A priori, the origin of formulae like $u = 2\, \partial_x^2 \ln \tau$ or (1.2) seems quite mysterious and unmotivated. Nevertheless, for a large class of integrable equations, they have a remarkable group theoretical interpretation within the, so called, tau-function approach pioneered by the japanese school (see for example [8]). Actually, this approach manifests the deep underlying connection of the integrable hierarchies of partial differential equations

with affine Kac-Moody algebras; a connection that is also apparent in the seminal work of Drinfel'd and Sokolov [9], where integrable hierarchies of equations are constructed in zero-curvature form.

The tau-function approach has been largely clarified by the work of Wilson [10,11] and of Kac and Wakimoto [12]. In this latter reference, the authors construct hierarchies of integrable equations directly in Hirota form associated to vertex operator representations of Kac-Moody algebras; then, the tau-functions describe the orbit of the highest-weight vector of the representation under the corresponding Kac-Moody group. On the other hand, the work of Wilson provides the group theoretical interpretation of the change of variables between the tau-functions and the natural variables in the zero curvature approach for several well known integrable equations like KdV and modified KdV [10], and non-linear Schrödinger [11] (see also [13]). In these articles, the change of variables is obtained by using a particular version of the well known dressing transformations of Zakharov and Shabat [14].

Using Wilson's ideas, the connection between the generalized Hirota equations of Kac and Wakimoto and the zero-curvature equations of [15] has been established in ref. [16]. It is worth noticing that the class of integrable equations of [15] is large enough to include practically all the generalizations of the Drinfel'd-Sokolov construction considered so far in the literature, and, therefore, it is desirable to have the tau-function description of all those integrable hierarchies of integrable equations. The reason why the results of [16] do not apply for all the integrable hierarchies of [15] is that the generalized Hirota equations of [12] are constructed in terms of level-one vertex operator representations of simply laced affine Kac-Moody algebras, while the integrable hierarchies of [15] require a definition of the tau-functions in terms of arbitrary highest-weight representations.

Another important restriction in the results of [16] is that they do not include the important class of integrable equations known as generalized Toda equations; *e.g.*, they do not explain the change of variables (1.2). Nevertheless, inspired by the results of [16], a definition for the tau-functions of the Toda equations has been proposed in [17].

The aim of this paper is to generalize the results of [16] and [17] in order to clarify the definition and relevance of tau-functions for a large class of integrable equations including both the integrable hierarchies of [15] and the non-abelian generalizations of the Toda equation. In our approach, the central role will be played by the dressing transformations in the manner described by Wilson [10,11]. This way, we will construct explicit solutions of certain non-linear integrable equations by dressing some "vacuum solutions". Actually, we will recognize the relevant equations by inspecting the properties of the resulting solutions.

This is reminiscent of the tau-function approach of [12], where the tau-functions are defined as the elements of the orbit of a highest-weight under the Kac-Moody group, and the generalized Hirota equations are just the equations characterizing those orbits; thus, the solutions and the equations are obtained simultaneously. In contrast, with our method we do not expect to produce all the solutions of the resulting equations, but only a subset that is conjectured to include the multi-soliton solutions.

The paper is organized as follows. In section 2 we describe the type of hierarchies we are going to consider, discuss their vacuum solutions and construct solutions using the dressing transformation method. In section 3 we define the tau-functions for all these hierarchies using integrable highest-weight representations of affine Kac-Moody algebras and generalizing some results known for level-one vertex operator representations. In section 4 we specialize our results to the generalized mKdV (and KdV) hierarchies of [15], and to the abelian and non abelian affine Toda theories. Conclusions are presented in section 5, and we also provide an appendix with our conventions about Kac-Moody algebras and their integrable highest-weight representations.

2. Vacuum solutions and dressing transformations.

Non-linear integrable hierarchies of equations are most conveniently discussed by associating them with a system of first-order differential equations

$$\mathcal{L}_N \Psi = 0\,, \tag{2.1}$$

where $\mathcal{L}_N$ are Lax operators of the form

$$\mathcal{L}_N \equiv \frac{\partial}{\partial t_N} - A_N \tag{2.2}$$

and the variables t_N are the various "times" of the hierarchy. Then, the equivalent zero-curvature formulation is obtained through the integrability conditions of the associated linear problem (2.1),

$$[\mathcal{L}_N\,,\,\mathcal{L}_M] = 0\,. \tag{2.3}$$

An equivalent way to express the relation between the solutions of the zero-curvature equations and of the associated linear problem is

$$A_N = \frac{\partial \Psi}{\partial t_N}\,\Psi^{-1}\,. \tag{2.4}$$

The class of integrable hierarchies of zero-curvature equations that will be studied here is constructed from graded Kac-Moody algebras in the following way (we have briefly summarized our conventions concerning Kac-Moody algebras in the appendix). Consider a complex affine Kac-Moody algebra $\mathfrak{g} = \widehat{g} \oplus \mathbb{C}\, d$ of rank r, and an integer gradation of its derived algebra $\widehat{g}$ labelled by a vector $\mathbf{s} = (s_0, s_1, \ldots, s_r)$ of $r+1$ non-negative co-prime integers such that

$$\widehat{g} = \bigoplus_{i\in\mathbb{Z}} \widehat{g}_{\,i}(\mathbf{s}) \quad \text{and} \quad [\,\widehat{g}_{\,i}(\mathbf{s}),\, \widehat{g}_{\,j}(\mathbf{s})] \subseteq \widehat{g}_{\,i+j}(\mathbf{s})\,. \tag{2.5}$$

We have in mind basically two types of integrable systems. The first one corresponds to the Generalized Drinfel'd-Sokolov Hierarchies considered in [15,16], which are generalizations of the KdV type hierarchies studied in [9]. In particular, and using the parlance of the original references, we will be interested in the generalized mKdV hierarchies, whose construction can be summarised as follows (see [15] and, especially, [16] for details). Given an integer gradation $\mathbf{s}$ of $\widehat{g}$ and a semisimple constant element E_l of grade l with respect to $\mathbf{s}$, one defines the Lax operator

$$L \equiv \partial_x + E_l + A\,, \tag{2.6}$$

where the components of A are the fields of the hierarchy. [1] They are functions of x and of the other times of the hierarchy taking values in the subspaces of $\widehat{g}$ with grades ranging from 0 to $l-1$. For each element in the centre of $\mathrm{Ker}(\mathrm{ad}\, E_l)$ with positive $\mathbf{s}$-grade N, one constructs a local functional of those fields, B_N, whose components take values in the subspaces $\widehat{g}_{\,0}(\mathbf{s}), \ldots, \widehat{g}_{\,N}(\mathbf{s})$. Then, B_N defines the flow equation

$$\frac{\partial L}{\partial t_N} = \big[B_N\,,\, L\big]\,, \tag{2.7}$$

and the resulting Lax operators $\mathcal{L}_N = \partial/\partial t_N - B_N$ commute among themselves [15].

The second type of integrable systems corresponds to the non-abelian affine Toda theories [18,17,19,20]. Given the integer gradation $\mathbf{s}$ of $\widehat{g}$, one chooses two constant elements $E_{\pm l}$ in $\widehat{g}_{\,\pm l}(\mathbf{s})$ and introduces two Lax operators

$$L_+ = \partial_+ - BF^+B^{-1}\,, \qquad L_- = \partial_- - \partial_- B\, B^{-1} + F^-\,. \tag{2.8}$$

[1] In [16] it was shown that the component of A along the central term of $\mathfrak{g}$ should not be considered as an actual degree of freedom of the hierarchy. This is the reason why these hierarchies can be equivalently formulated both in terms of affine Kac-Moody algebras or of the corresponding loop algebras (see section 4.1 for more details about this).

The field B is a function of $x_\pm$ taking values in the group obtained by exponentiating the zero graded subalgebra $\widehat{g}_0(\mathbf{s})$. As for the other fields, the functions $F^\pm$ can be decomposed as $F^\pm = E_{\pm l} + \sum_{m=1}^{l-1} F^\pm_m$, and F^+_m and F^-_m take values in $g_m(\mathbf{s})$ and $g_{-m}(\mathbf{s})$, respectively. Then, the condition $[L_+\,,\,L_-]=0$ provides the equations-of-motion of the theory, where $\partial_\pm$ are the derivatives with respect to the light-cone variables $x_\pm$. The well known abelian affine Toda equations are recovered with the principal gradation, $\mathbf{s}=(1,1,\ldots,1)$, and $l=1$. They possess an infinite number of conserved charges in involution [21], and these charges can be used to construct a hierarchy of integrable models through an infinite number of Lax operators that commute among themselves [22]. The non-abelian versions of the affine Toda equations are obtained with generic gradations $\mathbf{s}$ and $F^\pm_m = 0$ [18,17,20], while the most general case with $F^\pm_m \neq 0$ corresponds to the coupling of the latter systems with (spinor) matter fields [19].

An important common feature of all those hierarchies is that they possess trivial solutions which will be called "vacuum solutions". These particular solutions are singled out by the condition that the Lax operators evaluated on them lie on some abelian subalgebra of $\mathfrak{g}$, up to central terms. Then, the dressing transformation method can be used to generate an orbit of solutions out of each "vacuum". Moreover, it is generally conjectured that multi-soliton solutions lie in the resulting orbits. As a bonus, the fact that we only consider the particular subset of solutions connected with a generic vacuum allows one to perform the calculations in a very general way and, consequently, our results apply to a much broader class of hierarchies.

For a given choice of the Kac-Moody algebra $\mathfrak{g}$ and the gradation $\mathbf{s}$, let us consider Lax operators of the form (2.2) where the potentials can be decomposed as

$$A_N = \sum_{i=N_-}^{N_+} A_{N,i}\,, \quad \text{where} \quad A_{N,i} \in \widehat{g}_i(\mathbf{s}) \tag{2.9}$$

N_- and N_+ are non-positive and non-negative integers, respectively, and the times t_N are labelled by (positive or negative) integer numbers. The particular form of these potentials will be constrained only by the condition that the corresponding hierarchy admits vacuum solutions where they take the form

$$A_N^{(\mathrm{vac})} = \sum_{i=N_-}^{N_+} c_N^i b_i + \rho_N(t)\, c \;\equiv\; \varepsilon_N + \rho_N(t)\, c\,. \tag{2.10}$$

In this equation, c is the central element of $\widehat{g}$, and $b_i \in \widehat{g}_i(\mathbf{s})$ are the generators of a subalgebra $\widehat{s}$ of $\widehat{g}$ defined by

$$\widehat{s} = \{ b_i \in \widehat{g}_i(\mathbf{s})\,,\ i \in E \subset \mathbb{Z} \;\big|\; [b_i, b_j] = i\,\beta_i\, c\, \delta_{i+j,0} \}\,, \tag{2.11}$$

where β_i are arbitrary (vanishing or non-vanishing) complex numbers such that $\beta_{-i} = \beta_i$, and E is some set of integers numbers. Moreover, c_N^i are also arbitrary numbers, and $\rho_N(t)$ are $\mathbb{C}$-functions of the times t_N that satisfy the equations

$$\frac{\partial\, \rho_N(t)}{\partial\, t_M} - \frac{\partial\, \rho_M(t)}{\partial\, t_N} = \sum_i i\, \beta_i\, c_M^i\, c_N^{-i}\,. \tag{2.12}$$

These vacuum potentials correspond to the solution of the associated linear problem given by the group element (2.4)

$$\Psi^{(\mathrm{vac})} = \exp\Bigl(\sum_N \varepsilon_N t_N \;+\; \gamma(t)\, c\Bigr) \tag{2.13}$$

where the numeric function $\gamma(t)$ is a solution of the equations

$$\frac{\partial \gamma(t)}{\partial t_N} = \rho_N(t) + \frac{1}{2} \sum_{M,i} i\, \beta_i\, c_N^i\, c_M^{-i}\, t_M\,. \tag{2.14}$$

In terms of the associated linear problem, one can define an important set of transformations called “dressing transformations”, which take known solutions of the hierarchy to new solutions. Regarding the structure of the integrable hierarchies, these transformations have a deep meaning and, in fact, the group of dressing transformations can be viewed as the classical precursor of the quantum group symmetries [23]. Denote by $\widehat{G}_{\,-}(\mathbf{s})$, $\widehat{G}_{\,+}(\mathbf{s})$, and $\widehat{G}_{\,0}(\mathbf{s})$ the subgroups of the Kac-Moody group $\widehat{G}$ formed by exponentiating the subalgebras $\widehat{g}_{\,<0}(\mathbf{s}) \equiv \bigoplus_{i<0} \widehat{g}_{\,i}(\mathbf{s})$, $\widehat{g}_{\,>0}(\mathbf{s}) \equiv \bigoplus_{i>0} \widehat{g}_{\,i}(\mathbf{s})$, and $\widehat{g}_{\,0}(\mathbf{s})$, respectively. According to Wilson [10,11], the dressing transformations can be described in the following way. Consider a solution Ψ of the linear problem (2.1), and let $h = h_-\, h_0\, h_+$ be a constant element in the “big cell” of $\widehat{G}$, *i.e.*, in the subset $\widehat{G}_{\,-}(\mathbf{s})\; \widehat{G}_{\,0}(\mathbf{s})\; \widehat{G}_{\,+}(\mathbf{s})$ of $\widehat{G}$, such that

$$\Psi\, h\, \Psi^{-1} = (\Psi\, h\, \Psi^{-1})_-\; (\Psi\, h\, \Psi^{-1})_0\; (\Psi\, h\, \Psi^{-1})_+\,. \tag{2.15}$$

Notice that these conditions are equivalent to say that both h and $\Psi\, h\, \Psi^{-1}$ admit a generalized Gauss decomposition with respect to the gradation $\mathbf{s}$. Then

$$\begin{aligned} \Psi^h &= [(\Psi\, h\, \Psi^{-1})_-]^{-1}\, \Psi \\ &= (\Psi\, h\, \Psi^{-1})_0\, (\Psi\, h\, \Psi^{-1})_+\, \Psi\, h^{-1} \end{aligned} \tag{2.16}$$

is another solution of the linear problem. In order to prove it, introduce the notation $g_{0,\pm} \equiv (\Psi\, h\, \Psi^{-1})_{0,\pm}$ and $\partial_N \equiv \partial/\partial t_N$, and consider

$$\begin{aligned} \partial_N \Psi^h\, {\Psi^h}^{-1} &= -g_-^{-1}\, \partial_N g_- + g_-^{-1}(\partial_N \Psi\, \Psi^{-1}) g_- \\ &= \partial_N g_0\, g_0^{-1} + g_0 \partial_N g_+\, g_+^{-1} g_0^{-1} + g_0 g_+ (\partial_N \Psi\, \Psi^{-1}) g_+^{-1} g_0^{-1} \end{aligned} \tag{2.17}$$

Then, the first identity implies that $\partial_N \Psi^h \, {\Psi^h}^{-1} \in \bigoplus_{i\leq N_+} \widehat{g}_{\,i}(\mathbf{s})$, and the second that $\partial_N \Psi^h \, {\Psi^h}^{-1} \in \bigoplus_{i\geq N_-} \widehat{g}_{\,i}(\mathbf{s})$. Consequently

$$A_N^h = \frac{\partial \Psi^h}{\partial t_N} \, {\Psi^h}^{-1} \in \bigoplus_{i=N_-}^{N_+} \widehat{g}_{\,i}(\mathbf{s}) \,, \tag{2.18}$$

and, taking into account (2.9), it is a solution of the hierarchy of zero-curvature equations. [2]

For any h lying in the big cell of $\widehat{G}$, the transformation

$$\mathcal{D}_h : \Psi \;\mapsto\; \Psi^h \,, \quad \text{or} \quad A_N \mapsto A_N^h \,, \tag{2.19}$$

is called a dressing transformation, and an important property is that their composition law follows just from the composition law of $\widehat{G}$, *i.e.*, $\mathcal{D}_g \circ \mathcal{D}_h = \mathcal{D}_{gh}$. [3]

Now, the orbit of the vacuum solution (2.13) under the group of dressing transformations can be easily constructed using eqs. (2.16) and (2.18). For any element h of the big cell of $\widehat{G}$, let us define

$$\begin{aligned} &\Theta^{-1} = (\Psi^{(\mathrm{vac})}\, h \,{\Psi^{(\mathrm{vac})}}^{-1})_- \,, \qquad B^{-1} = (\Psi^{(\mathrm{vac})}\, h \,{\Psi^{(\mathrm{vac})}}^{-1})_0 \,, \\ &\Upsilon = (\Psi^{(\mathrm{vac})}\, h \,{\Psi^{(\mathrm{vac})}}^{-1})_+ \,, \quad \text{and} \quad \Omega = B^{-1}\, \Upsilon \,. \end{aligned} \tag{2.20}$$

Then, under the dressing transformation generated by h,

$$\Psi^{(\mathrm{vac})} \;\mapsto\; \Psi^h = \Theta\, \Psi^{(\mathrm{vac})} = \Omega\, \Psi^{(\mathrm{vac})}\, h^{-1} \,, \tag{2.21}$$

or, equivalently, $A_N^{(\mathrm{vac})}$ becomes

$$\begin{aligned} A_N^h - \rho_N(t)\, c \;&=\; \Theta\, \varepsilon_N\, \Theta^{-1} \;+\; \partial_N \Theta\, \Theta^{-1} \;\in\; \bigoplus_{i\leq N_+} \widehat{g}_{\,i}(\mathbf{s}) \\ &=\; \Omega\, \varepsilon_N\, \Omega^{-1} \;+\; \partial_N \Omega\, \Omega^{-1} \;\in\; \bigoplus_{i\geq N_-} \widehat{g}_{\,i}(\mathbf{s}), \end{aligned} \tag{2.22}$$

Eqs. (2.20) and (2.22) summarize the outcome of the dressing transformation method, which, starting with some vacuum solution (2.10), associates a solution of the zero-curvature equations (2.3) to each constant element h in the big cell of $\widehat{G}$. The construction of this solution involves two steps. First, the eqs. (2.22) can be understood as a local

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[2] If the fields of the hierarchy are such that $A_{N,i}$ does not span the whole subspace $\widehat{g}_{\,i}(\mathbf{s})$ then we have to impose further constraints on the group elements performing the dressing transformation (see section 4.2).

[3] The dressing transformations of [23] are defined in a different way through $\mathcal{D}_h^*(\Psi) = [(\Psi h \Psi^{-1})_-]^{-1} \Psi h_-$, which leads to a different composition law characterised by a classical r-matrix. Nevertheless, this definition induces exactly the same transformation $A_N \mapsto A_N^h$ among the solutions of the zero-curvature equations.

change of variables between the components of the potential A_N and some components of the group elements Θ, B and Υ.

The second step consists in obtaining the value of the required components of Θ, B and Υ from eq. (2.20). This is usually done by considering matrix elements of the form

$$\langle\mu|\, \Theta^{-1}\, B^{-1}\, \Upsilon\, |\mu'\rangle = \langle\mu|\, e^{\sum_N \varepsilon_N t_N}\, h\, e^{-\sum_N \varepsilon_N t_N}\, |\mu'\rangle\,, \tag{2.23}$$

where $|\mu\rangle$ and $|\mu'\rangle$ are vectors in a given representation of $\mathfrak{g}$. The appropriate set of vectors is specified by the condition that all the required components of Θ, B and Υ can be expressed in terms of the resulting matrix elements. It will be show in the next section that the required matrix elements, considered as functions of the group element h, constitute the generalization of the Hirota's tau-functions for these hierarchies. Moreover, eq. (2.23) is the analogue of the, so called, solitonic specialization of the Leznov-Saveliev solution proposed in [24,25,26,17,19] for the affine (abelian and non-abelian) Toda theories.

Consider now the common eigenvectors of the adjoint action of the ε_N's that specify the vacuum solution (2.10). Then, the important class of multi-soliton solutions is conjectured to correspond to group elements h which are the product of exponentials of eigenvectors

$$h = e^{F_1}\, e^{F_2}\, \ldots e^{F_n}\,, \qquad [\varepsilon_N\,,\, F_k] = \omega_N^{(k)}\, F_k\,, \quad k = 1,2,\ldots n\,. \tag{2.24}$$

In this case, the dependence of the solution upon the times t_N can be made quite explicit

$$\langle\mu|\, \Theta^{-1}\, B^{-1}\, \Upsilon\, |\mu'\rangle = \langle\mu|\, \prod_{k=1}^{n} \exp(e^{\sum_N \omega_N^{(k)} t_N} F_k)\, |\mu'\rangle\,. \tag{2.25}$$

The conjecture that multi-soliton solutions are associated with group elements of the form (2.24) naturally follows from the well known properties of the multi-soliton solutions of affine Toda equations and of hierarchies of the KdV type, and, in the sine-Gordon theory, it has been explicitly checked in ref. [27]. Actually, in all these cases, the multi-soliton solutions are obtained in terms of representations of the "vertex operator" type where the corresponding eigenvectors are nilpotent. Then, for each eigenvector F_k there exists a positive integer number m_k such that $(F_k)^m \neq 0$ only if $m \leq m_k$. This remarkable property simplifies the form of (2.25) because it implies that $e^{F_k} = 1 + F_k + \cdots + (F_k)^{m_k}/m_k!$, which provides a group-theoretical justification of Hirota's method.

An interesting feature of the dressing transformations method is the possibility of relating the solutions of different integrable equations. Consider two different integrable hierarchies whose vacuum solutions are compatible, in the sense that the corresponding

vacuum Lax operators commute. Then, one can consider the original integrable equations as the restriction of a larger hierarchy of equations. Consequently, the solutions obtained through the group of dressing transformations can also be understood in terms of the solutions of the larger hierarchy, which implies certain relations among them. We will show in section 4 that this possibility generalizes the well knonw relation between the solutions of the mKdV and sine-Gordon equations.

3. The tau-functions

According to the discussion in the previous section, the orbits generated by the group of dressing transformations acting on some vacuum provide solutions of certain integrable hierarchies of equations. Making contact with the method of Hirota, the generalized "tau-functions" that will be defined in this section constitute a new set of variables to describe those solutions. One of the characteristic properties of these variables is that they substantially simplify the task of constructing multi-soliton solutions [17]. The group-theoretical interpretation of this property has already been pointed out in the previous section. Tau-functions are given by certain matrix elements in a appropriate representation of the Kac-Moody Group $\widehat{G}$. Moreover, the tau-functions corresponding to the multi-soliton solutions are expected to involve nilpotent elements of $\widehat{G}$, which is the origin of their remarkable simple form.

The tau-function formulation of the Generalized Drinfel'd-Sokolov Hierarchies of [15] has already been worked out in [16], which, in fact, has largely inspired our approach. However, there are two important differences between our results and those of [16]. Firstly, our approach applies to the affine Toda equations too, and, secondly, it does not rely upon the use of (level-one) vertex operator representations.

At this point, it is worth recalling that the solutions constructed in section 2 are completely representation-independent. In contrast, our definition of tau-functions makes use of a special class of representations of the Kac-Moody algebra $\widehat{g}$ called "integrable highest-weight" representations, which are briefly reviewed in the appendix. The reason why these representations are called "integrable" is the following. For an infinite-dimensional representation, it is generally not possible to go from a representation of the algebra $\widehat{g}$ to a representation of the corresponding group $\widehat{G}$ via the exponential map $x \mapsto \mathrm{e}^{x}$. However, the construction does work if, for instance, the formal power series terminates at a certain power of x, or if the representation space admits a basis of eigenvalues of x.

These conditions, applied to the Chevalley generators of $\widehat{g}$, single out this special type of representations.

The generalized tau-functions will be sets of matrix elements of the form indicated on the right-hand-side of (2.23), considered as functions of the group element h. They are characterized by the condition that they allow one to parameterize all the components of Θ, B, and Υ required to specify the solutions (2.22) of the zero-curvature equations (2.3). As we have discussed before, the tau-functions corresponding to the multi-soliton solutions are expected to have a very simple form. However, in contrast with the original method of Hirota, we cannot ensure in general that the equations of the hierarchy become simpler in terms of this new set of variables.

First, let us discuss the generalized Hirota tau-functions associated with the components of B. In equation (2.23), these components can be isolated by considering the vectors $|\mu_0\rangle$ of an integrable highest-weight representation $L(\tilde{\mathbf{s}})$ of $\mathfrak{g}$ which are annihilated by all the elements in $\widehat{g}_{>0}(\mathbf{s})$, *i.e.*, $T|\mu_0\rangle = 0$ and $\langle\mu_0|T' = 0$ for all $T \in \widehat{g}_{>0}(\mathbf{s})$ and $T' \in \widehat{g}_{<0}(\mathbf{s})$, respectively. Then, the corresponding tau-functions are defined as [4]

$$\begin{aligned} \tau_{\mu_0,\mu_0'}(t) &= \langle\mu_0'|\Psi^{(\mathrm{vac})}\, h\, {\Psi^{(\mathrm{vac})}}^{-1}|\mu_0\rangle \\ &= \langle\mu_0'|\, e^{\sum_N \varepsilon_N t_N}\, h\, e^{-\sum_N \varepsilon_N t_N}\, |\mu_0\rangle\,, \end{aligned} \tag{3.1}$$

and, in terms of them, equation (2.23) becomes just

$$\langle\mu_0'|\, B^{-1}\, |\mu_0\rangle \;=\; \tau_{\mu_0,\mu_0'}(t)\,. \tag{3.2}$$

By construction, $\widehat{g}_0(\mathbf{s})$ always contains the central element c of the Kac-Moody algebra, but it is always possible to split the contribution of the corresponding field in (3.2). Let $s_q \neq 0$ and consider the subalgebra $\mathring{\mathfrak{g}}$ of $\mathfrak{g}$ generated by the $e_i^{\pm}$ with $i = 0, \ldots, r$ but $i \neq q$, which is a semisimple finite Lie algebra of rank r ($\mathring{\mathfrak{g}}$ is always simple if $q = 0$). Then, $\widehat{g}_0(\mathbf{s}) = \big(\widehat{g}_0(\mathbf{s}) \cap \mathring{\mathfrak{g}}\big) \oplus \mathbb{C}\, c$ and, correspondingly, B can be split as $B = b\, \exp(\nu\, c)$. Here, ν is the field along c, and b is a function taking values in the semisimple finite Lie group $\mathring{G}_0$ whose Lie algebra is $\widehat{g}_0(\mathbf{s}) \cap \mathring{\mathfrak{g}}$. Since $\tilde{K} = \sum_{i=0}^{r} k_i^{\vee}\, \tilde{s}_i$ is the level of the representation $L(\tilde{\mathbf{s}})$, eq. (3.2) is equivalent to

$$\langle\mu_0'|\, B^{-1}\, |\mu_0\rangle \;=\; \mathrm{e}^{-\nu\,\tilde{K}}\, \langle\mu_0'|\, b^{-1}\, |\mu_0\rangle \;=\; \tau_{\mu_0,\mu_0'}(t)\,. \tag{3.3}$$

[4] Since the resulting relations between tau-functions and components of the A_N's will be considered as generic changes of variables (see (1.1) and (1.2)), we will not generally indicate the intrinsic dependence of the tau-functions on the group element h.

Moreover, it is always possible to introduce a tau-function for the field ν. Let us consider the highest-weight vector $|v_q\rangle$ of the fundamental representation $L(q)$, which is obviously annihilated by all the elements in $\mathring{\mathfrak{g}}$. Therefore,

$$\langle v_q|\, B^{-1}\, |v_q\rangle \;=\; \mathrm{e}^{-\nu\, k_q^\vee} \;= \tau_{v_q,v_q}(t) \equiv\; \tau_q^{(0)}(t)\,, \tag{3.4}$$

which leads to

$$\langle \mu_0'|\, b^{-1}\, |\mu_0\rangle \;=\; \frac{\tau_{\mu_0,\mu_0'}(t)}{\left(\tau_q^{(0)}(t)\right)^{\tilde{K}/k_q^\vee}} \quad \text{and} \quad \nu \;=\; -\;\ln \frac{\tau_q^{(0)}(t)}{k_q^\vee}\,. \tag{3.5}$$

Finally, recall that the vectors $|\mu_0\rangle$ form a representation of the semisimple Lie group $\mathring{G}_0$. Therefore, if $L(\tilde{\mathbf{s}})$ is chosen such that this representation is faithful, eq. (3.5) allows one to obtain all the components of b in terms of the generalized tau-functions $\tau_{\mu_0,\mu_0'}$ and $\tau_q^{(0)}$. Notice that, in this case, the definition of generalized tau-functions coincide exactly with the quantities involved in the solitonic specialization of the Leznov-Saveliev solution proposed in [26].

Let us now discuss the generalized tau-functions associated with the components of Θ. Consider the gradation $\mathbf{s}$ of $\mathfrak{g}$ involved in the definition of the integrable hierarchy. For each $s_i \neq 0$, let us consider the highest-weight vector of the fundamental representation $L(i)$ and define the (right) tau-function vector

$$\begin{aligned} |\tau_i^R(t)\rangle \;&=\; \Psi^{(\mathrm{vac})}\, h\, \Psi^{(\mathrm{vac})^{-1}}\, |v_i\rangle \\ &=\; e^{\sum_N \varepsilon_N t_N}\, h\, e^{-\sum_N \varepsilon_N t_N}\, |v_i\rangle \end{aligned} \tag{3.6}$$

.

Notice that $|\tau_i^R(t)\rangle$ is a vector in the representation $L(i)$. Therefore, it has infinite components, and it will be shown soon that the role of the Hirota tau-functions will be played by a finite subset of them. Taking into account that $|v_i\rangle$ is annihilated by all the elements in $g_{>0}(\mathbf{s})$, equation (2.23) implies

$$\Theta^{-1}\, B^{-1}\, |v_i\rangle \;=\; |\tau_i^R(t)\rangle\,, \quad i=0,\ldots,r \quad \text{and} \quad s_i \neq 0\,. \tag{3.7}$$

The definition (3.6) is inspired by the tau-function approach of [12,16,17]. However, in [16,17], the authors consider a unique tau-function $|\tau_{\mathbf{s}}(t)\rangle \in L(\mathbf{s})$. In fact, one could equally consider different tau-functions $|\tau_{\mathbf{s}'}(t)\rangle$ associated with any integrable representation $L(\mathbf{s}')$ such that $s_i' \neq 0$ if, and only if, $s_i \neq 0$. According to eq. (A.6), all these choices lead to the same results, but ours is the most economical.

Since, for any integrable representation, the derivation $d_{\mathbf{s}}$ can be diagonalized acting on $L(\mathbf{s})$, these tau-functions vectors can be decomposed as

$$|\tau_i^R(t)\rangle \;=\; \sum_{-j\in\mathbb{Z}_{\leq 0}} |\tau_i^{R(-j)}(t)\rangle\,, \qquad d_i\,|\tau_i^{R(-j)}(t)\rangle \;=\; -j\,|\tau_i^{R(-j)}(t)\rangle\,, \tag{3.8}$$

where we have used that $\Theta\in\,\widehat{G}_{<0}(\mathbf{s})$ and $B\in\,\widehat{G}_{0}(\mathbf{s})$, and d_i indicates the derivation corresponding to the gradation with $s_j=\delta_{j,i}$ (see the appendix). Moreover, the highest-weight vector is an eigenvector of the subalgebra $\widehat{g}_{0}(\mathbf{s})$ and, consequently, of B. Therefore,

$$|\tau_i^{R(0)}(t)\rangle \;=\; B^{-1}\,|v_i\rangle \;=\; \tau_i^{(0)}(t)\,|v_i\rangle\,, \tag{3.9}$$

where, $\tau_i^{(0)}(t)$ is a $\mathbb{C}$-function, not a vector of $L(i)$, whose definition is [5]

$$\tau_i^{(0)}(t) \;=\; \langle v_i|\, e^{\sum_N \varepsilon_N t_N}\; h\; e^{-\sum_N \varepsilon_N t_N}\,|v_i\rangle \;\equiv\; \tau_{v_i,v_i}(t) \tag{3.10}$$

(compare with eq. (3.4)). Therefore, eq. (3.7) becomes

$$\Theta^{-1}\,|v_i\rangle \;=\; \frac{1}{\tau_i^{(0)}(t)}\,|\tau_i^R(t)\rangle\,, \tag{3.11}$$

which is the generalization of the eq. (5.1) of [16] for general integrable highest-weight representations of $\mathfrak{g}$. Eq. (3.11) allows one to express all the components of Θ in terms of the components of $|\tau_i^R(t)\rangle$ for all $i=0,\ldots,r$ with $s_i\neq 0$ (for instance, by using the positive definite Hermitian form of $L(i)$). However, it is obvious that only a finite subset of them enter in the definition of the potentials A_N through eq. (2.22).

In exactly the same way, one can introduce another set of "left" tau-function vectors through

$$\langle\tau_i^L(t)| \;=\; \langle v_i|\,\Psi^{(\mathrm{vac})}\,h\,{\Psi^{(\mathrm{vac})}}^{-1}\,, \tag{3.12}$$

which leads to

$$\langle v_i|\,\Upsilon \;=\; \langle\tau_i^L(t)|\,\frac{1}{\tau_i^{(0)}(t)}\,, \tag{3.13}$$

and allows one to express all the components of Υ in terms of the components of $\langle\tau_i^L(t)|$ for all $i=0,\ldots,r$ with $s_i\neq 0$.

Summarising, the generalized Hirota tau-functions of these hierarchies consist of the subset of functions $\tau_{\mu_0,\mu_0'}$ and of components of $|\tau_i^R\rangle$ and $\langle\tau_i^L|$ required to parameterize all

[5] To compare with (3.2), notice that $|\mu_0\rangle=|v_i\rangle$ forms a one-dimensional representation of $\widehat{g}_{0}(\mathbf{s})$ and, consequently, $\tau_{v_i,\mu_0'}(t)$ vanishes unless $|\mu_0'\rangle=|v_i\rangle$. Therefore, for non-abelian G_0, the required tau-functions $\tau_{\mu_0,\mu_0'}(t)$ have to involve the fundamental integrable representations $L(j)$ corresponding to $s_j=0$, in contrast with $|\tau_i^R(t)\rangle$ (see eq. (3.7)).

the components of the potentials A_N in eq. (2.22). Then, for the multi-soliton solutions corresponding to the group element h specified in (2.24), their truncated power series expansion follows from the possible nilpotency of the eigenvectors F_k in these representations. For instance, if $n = 1$ in (2.24) and $F_1^m|\mu_0\rangle = F_1^m|v_i\rangle = 0$ unless $m \leq m_1$, then

$$
\begin{aligned}
\tau_{\mu_0,\mu_0'}(t) &= \tilde{\tau}^0_{\mu_0,\mu_0'} + \tilde{\tau}^1_{\mu_0,\mu_0'} + \ldots + \tilde{\tau}^{m_1}_{\mu_0,\mu_0'} \\
&= \sum_{k=0}^{m_1} \frac{1}{k!}\, \mathrm{e}^{\,k\,\sum_N \omega_N t_N}\, \langle \mu_0' |\, F_1^k\, | \mu_0 \rangle \,, \quad \text{and} \\
|\tau_i^R(t)\rangle &= \sum_{k=0}^{m_1} \frac{1}{k!}\, \mathrm{e}^{\,k\,\sum_N \omega_N t_N}\, F_1^k\, | v_i \rangle \,.
\end{aligned} \tag{3.14}
$$

4. Examples

The orbits generated by the group of dressing transformations acting on the vacuum configurations described in section 2 provide solutions of the generalized mKdV equations of [15], and of the non-abelian affine Toda equations. In this section we will characterize the appropriated choices for $\widehat{s}$, and derive the relation between the original variables and their tau-functions in the simplest cases in order to illustrate the main issues of our formalism. Moreover, these examples show how the usual definitions of tau-functions in abelian Toda equations [3,4] are precisely recovered.

For simplicity, we will restrict ourselves to vacuum solutions associated with untwisted affine Kac-Moody algebras, although our construction applies also to the twisted case. Then, it will be convenient to use the realization of $\widehat{g}$ as the central extension of the loop algebra of simple finite Lie algebra g, such that

$$
\widehat{g} = \left\{ u^{(m)} \;\middle|\; u \in g\,,\ m \in \mathbb{Z} \right\} \oplus \mathbb{C}\, c\,, \qquad \mathfrak{g} \equiv g^{(1)} = \widehat{g} \oplus \mathbb{C}\, d\,,
$$

$$
\begin{aligned}
\left[u^{(m)}\,,\, v^{(n)} \right] &= [u\,,\, v]^{(m+n)} + m\, \mathrm{Tr}(u\, v)\, c\, \delta_{m+n,0}\,, \\
\left[d\,,\, u^{(m)} \right] &= m\, u^{(m)}\,, \qquad \left[c\,,\, d \right] = \left[c\,,\, u^{(m)} \right] = 0\,,
\end{aligned} \tag{4.1}
$$

where $\mathrm{Tr}(\cdot\,\cdot)$ denotes the Cartan-Killing form of g. Then, the Chevalley generators of $g^{(1)}$ are

$$
e_i^{\pm} = \begin{cases} E^{(0)}_{\pm\boldsymbol{\alpha}_i}\,, & \text{for } i = 1, \ldots, r\,, \\[2ex] E^{(\pm 1)}_{\pm\boldsymbol{\alpha}_0}\,, & \text{for } i = 0\,, \end{cases} \qquad h_i = \frac{2}{\boldsymbol{\alpha}_i^2}\, \boldsymbol{\alpha}_i \cdot \boldsymbol{H}^{(0)} + c\, \delta_{i,0}\,, \tag{4.2}
$$

where $\boldsymbol{\alpha}_0 = -\sum_{i=1}^{r} k_i\, \boldsymbol{\alpha}_i$ is minus the highest root of g normalized as $\boldsymbol{\alpha}_0^2 = 2$, $E_{\boldsymbol{\alpha}}$ is the step operator of the root $\boldsymbol{\alpha}$, and $\boldsymbol{H}$ is an element of the Cartan subalgebra of g ($\boldsymbol{H}$ and $\boldsymbol{\alpha}$ live in the same r-dimensional vector space).

We will also use the notation

$$\widehat{g}_{\leq k}(\mathbf{s}) = \bigoplus_{i\leq k} \widehat{g}_{i}(\mathbf{s}) \,, \qquad \widehat{g}_{\geq k}(\mathbf{s}) = \bigoplus_{i\geq k} \widehat{g}_{i}(\mathbf{s}) \,, \tag{4.3}$$

and denote by $\mathrm{P}_{\geq k[\mathbf{s}]}$ and $\mathrm{P}_{<k[\mathbf{s}]}$ the projectors onto $\widehat{g}_{\geq k}(\mathbf{s})$ and $\widehat{g}_{<k}(\mathbf{s})$, respectively.

Different choices of the subset $\widehat{s}$ introduced in (2.11) lead to solutions of different integrable hierarchies. However, particularly interesting vacuum solutions arise when $\widehat{s}$ is a subset of a Heisenberg subalgebra of $\mathfrak{g}$, which, up to the central element c, are special types of maximally commuting subalgebras whose precise definition can be found in [28]. In particular, when $\mathfrak{g} = g^{(1)}$, they correspond to the affinization of a Cartan subalgebra of g by means of an inner automorphism. This implies that the inequivalent Heisenberg subalgebras of $g^{(1)}$ are classified by the conjugacy classes of the Weyl group of g [28]. Consequently, their structure is

$$\mathcal{H}[w] = \mathbb{C}\, c + \sum_{i\in I_{[w]}+\mathbb{Z}N_{[w]}} \mathbb{C}\Lambda_i \,, \qquad [\Lambda_i, \Lambda_j] = i\, c\, \delta_{i+j,0} \,, \tag{4.4}$$

where $[w]$ denotes a conjugacy class of the Weyl group of g, and $I_{[w]}$ is a set of r integers ≥ 0 and $< N_{[w]}$. The set $\{c,\, \Lambda_i \bigm| i \in I_{[w]} + \mathbb{Z}N_{[w]}\}$ is a basis of $\mathcal{H}[w]$ whose elements are graded with respect to the associated $[w]$-dependent gradation $\mathbf{s}^w = (s_0^w, \ldots, s_r^w)$. The gradation $\mathbf{s}^w$ fixes the set $I_{[w]}$ and the integer $N_{[w]} = \sum_{i=0}^{r} k_i s_i^w$, where $k_0 = 1$, $k_1, \ldots, k_r$ are the labels of the extended Dynkin diagram of g, which also specify its highest root $\boldsymbol{\alpha}_0$.

4.1 *Generalized mKdV and KdV hierarchies.*

Let Λ_i be an element of a Heisenberg subalgebra $\mathcal{H}[w]$ of $g^{(1)}$ whose $\mathbf{s}^w$-grade is $i > 0$, and consider the subalgebra

$$\widehat{s} = \mathrm{Cent}\,\big(\mathrm{Ker}(\mathrm{ad}\,\Lambda_i)\big) \cap \widehat{g}_{\geq 0}(\mathbf{s}^w) \subseteq \mathcal{H}[w] \cap \widehat{g}_{\geq 0}(\mathbf{s}^w) \,, \tag{4.5}$$

where by Cent $(\bullet)$ we mean the subalgebra of $(\bullet)$ generated by the elements which commute, up to the central element c, with all elements of $(\bullet)$. Then, $\widehat{s}$ gives rise to the vacuum solution

$$A_N^{(\mathrm{vac})} \;=\; \Lambda_N \;+\; \rho_N(t)\, c \,, \tag{4.6}$$

which is labelled by the set of integers N such that $\widehat{s} \cap \widehat{g}_{N}(\mathbf{s}^w)$ is not empty, and where $\Lambda_N \in \widehat{s} \cap \widehat{g}_{N}(\mathbf{s}^w)$. To compare with eq. (2.10), $N_- = 0$, $N_+ = N$, and $c_N^j = \delta_{j,N}$.

According with eq. (2.22), the orbit of solutions generated by the group of dressing transformations acting on this vacuum consists of the Lax operators

$$\begin{aligned} \mathcal{L}_N^h &= \Theta\big(\frac{\partial}{\partial t_N} - \Lambda_N\big)\Theta^{-1} - \rho_N(t)\, c \\ &= \frac{\partial}{\partial t_N} - \mathrm{P}_{\geq 0[\mathbf{s}^w]}\big(\Theta\, \Lambda_N\, \Theta^{-1}\big) - \rho_N(t)\, c\,, \end{aligned} \tag{4.7}$$

where Θ is defined in (2.20) and it will be understood as a function of the group element h.

Then, the eqs. (4.7) provide solutions for one of the generalized Drinfel'd-Sokolov hierarchies of [15] (see also [16]). In particular, for the generalized mKdV hierarchy associated with the Lax operator

$$L = \frac{\partial}{\partial x} - \Lambda - \widetilde{q} \equiv \mathcal{L}_i\,, \tag{4.8}$$

where $\Lambda = \Lambda_i$, $x = t_i$, and

$$\widetilde{q} \in \widehat{g}_{\geq 0}(\mathbf{s}^w) \cap \widehat{g}_{<i}(\mathbf{s}^w)\,. \tag{4.9}$$

In order to prove it, let us briefly summarize its construction. It is based on the existence of a unique transformation such that

$$\Phi\, L\, \Phi^{-1} = \frac{\partial}{\partial x} - \Lambda - h\,, \tag{4.10}$$

where $\Phi = \exp y$ with $y \in \mathrm{Im}(\mathrm{ad}\,\Lambda) \cap \widehat{g}_{<0}(\mathbf{s}^w)$, and $h \in \mathrm{Ker}(\mathrm{ad}\,\Lambda) \cap \widehat{g}_{<i}(\mathbf{s}^w)$ are local functionals of the components of $\widetilde{q}$ and their x-derivatives (notice that $\mathrm{Ker}(\mathrm{ad}\,\Lambda)$ is non-abelian in general). There is a difference between the situation in [15] and the situation in [16] and here. Here, $\widetilde{q}$ may have a component along the central element of $g^{(1)}$, say $\widetilde{q} = q + q_c\, c$, where q is the component of $\widetilde{q}$ in the loop algebra of g. Then, it is straightforward to show that Φ and $h - q_c\, c$ depend only on the components of q. The hierarchy consists of an infinite set of commuting flows associated with the elements Λ_N in $\widehat{s}$, and they are defined by the zero-curvature equations

$$\big[\frac{\partial}{\partial t_N} - \mathrm{P}_{\geq 0[\mathbf{s}^w]}(\Phi^{-1}\, \Lambda_N\, \Phi)\,,\, L\big] = 0\,. \tag{4.11}$$

Moreover, since Φ is a differential polynomial of q and it does not depend on q_c, this flow equation induces a flow equation for q which can be written in the form

$$\frac{\partial q}{\partial t_j} = F_j\big(q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2}, \ldots\big)\,, \tag{4.12}$$

for some polynomial functions F_j. In contrast, the corresponding equation for the component along the central element of $g^{(1)}$ is $\partial_N q_c = -\partial_x(\Phi^{-1}\, \Lambda_N\, \Phi)_c$, where $(\bullet)_c$ is the

component of $\bullet$ along $\mathbb{C}\, c$. Since Φ depends only on q, this shows that q_c is not a real degree of freedom, which is the reason why these hierarchies can be associated both with Kac-Moody or loop algebras.

Let us consider the mKdV hierarchy (4.11) constrained by the condition that $h \in \operatorname{Ker}(\operatorname{ad}\Lambda) \cap \widehat{g}_{<0}(\mathbf{s}^w)$. Since all the vanishing components of h in $\widehat{g}_{\geq 0}(\mathbf{s}^w)$ are functionals of $\widetilde{q}$, this implies a constraint on the mKdV field, and it is easy to check that it is compatible with the flow equations. Then, using (4.10) and introducing a non-local functional χ of q such that $\chi \in \exp\bigl(\operatorname{Ker}(\operatorname{ad}\Lambda) \cap \widehat{g}_{<0}(\mathbf{s}^w)\bigr)$ and $\partial_x \chi\, \chi^{-1} = h$, the Lax operator becomes

$$\begin{aligned} L &= \Theta\bigl(\frac{\partial}{\partial x} - \Lambda\bigr)\Theta^{-1} + (\chi\Lambda\chi^{-1})_c \\ &= \frac{\partial}{\partial x} - \mathrm{P}_{\geq 0[\mathbf{s}^w]}\bigl(\Theta\Lambda\Theta^{-1}\bigr) + (\chi\Lambda\chi^{-1})_c\,, \end{aligned} \tag{4.13}$$

where $\Theta = \Phi^{-1}\,\chi \in \widehat{G}_-(\mathbf{s}^w)$. Moreover, the Lax operators that define the flows of the hierarchy can be written as

$$\frac{\partial}{\partial t_N} - \mathrm{P}_{\geq 0[\mathbf{s}^w]}(\Phi^{-1}\,\Lambda_N\,\Phi) = \frac{\partial}{\partial t_N} - \mathrm{P}_{\geq 0[\mathbf{s}^w]}(\Theta\,\Lambda_N\,\Theta^{-1}) - (\chi^{-1}\,\Lambda_N\,\chi)_c\,. \tag{4.14}$$

This, compared with (4.7), shows that the orbit generated by the group of dressing transformations acting on the vacuum solution (4.6) actually consists of solutions of the generalized mKdV hierarchy associated with the Lax operator (4.8).

Eqs. (4.8) and (4.13) provide the change of variables between $\widetilde{q}$ and the components of Θ, and they show that only the finite number of terms of Θ with $\mathbf{s}^w$-grade ranging from $-i$ to -1 are required. Therefore, in this case, the generalized tau-functions correspond to a finite set of components of the vectors $|\tau_i^R(t)\rangle$ in the fundamental integrable representations $L(i)$ such that $s_i^w \neq 0$. Let us remark that this case is covered by the results of [16] only if these representations are of level-one, which means that that $s_i^w \neq 0$ only if $k_i^\vee = 1$.

As an specific example, let us discuss the Drinfel'd-Sokolov generalized mKdV hierarchies associated to a simple finite Lie algebra g. They are recovered from the principal Heisenberg subalgebra, which is graded with respect to the principal gradation $\mathbf{s}_{\mathrm{p}} = (1, 1, \ldots, 1)$, and the Lax operator (4.8) where, in this case,

$$\Lambda = \Lambda_1 = \sum_{i=0}^{r} e_i^+ \quad \text{and} \quad \widetilde{q} = \sum_{i=0}^{r} q_i\, h_i\,. \tag{4.15}$$

The change of variables between $\widetilde{q}$ and the components of Θ follows from (4.8) and (4.13) by writing

$$\Theta = \exp\left(\theta_{-1} + \cdots\right) \in \widehat{G}_-(\mathbf{s}_p)\,,$$

with $\theta_{-1} \in \widehat{g}_{-1}(\mathbf{s}_p)$, which leads to

$$\widetilde{q} = [\theta_{-1}, \Lambda] . \tag{4.16}$$

The relation between θ_{-1} and the tau-functions has to be obtained from eq. (3.11). Consider the decomposition $\theta_{-1} = \sum_{i=0}^{r} a_i e_i^-$ for some functions a_j, then (3.11) implies that

$$\sum_{i=0}^{r} a_i\, e_i^- \, |v_j\rangle = - \frac{1}{\tau_j^{(0)}(t)} \, |\tau_j^{R(-1)}(t)\rangle . \tag{4.17}$$

Moreover, since the times are labelled by positive integers (the exponents of g plus its Coxeter number times some non-negative integer) and Λ_N annihilates the highest-weight vectors for $N > 0$, eq. (3.6) reduces to

$$|\tau_j^R(t)\rangle = \mathrm{e}^{\sum_N t_N \Lambda_N} \, h \, |v_j\rangle , \tag{4.18}$$

and eq. (3.10) becomes

$$\tau_j^{(0)}(t) = \langle v_j| \, \mathrm{e}^{\sum_N t_N \Lambda_N} \, h \, |v_j\rangle . \tag{4.19}$$

Then, eqs. (4.17) and (4.18) imply that

$$a_j = - \frac{1}{\tau_j^{(0)}(t)} \, \langle v_j| \, e_j^+ \, \mathrm{e}^{\sum_N t_N \Lambda_N} \, h \, |v_j\rangle , \tag{4.20}$$

but $\langle v_j| \, e_j^+ = \langle v_j|\Lambda_1$, which, taking into account (4.19), finally leads to

$$a_j = - \frac{\partial}{\partial x} \ln \tau_j^{(0)}(t) . \tag{4.21}$$

Therefore, using (4.16), the resulting change of variables is

$$\begin{aligned} \widetilde{q} &= \sum_{i=0}^{r} \frac{\partial}{\partial x} \ln \tau_i^{(0)}(t) \, h_i \\ &= \sum_{i=1}^{r} \frac{\partial}{\partial x} \ln\left(\frac{\tau_i^{(0)}(t)}{\tau_0^{(0)\, k_i^\vee}(t)} \right) \frac{2}{\boldsymbol{\alpha}_i^2} \, \boldsymbol{\alpha}_i \cdot \boldsymbol{H}^{(0)} + \tau_0^{(0)}(t) \, c , \end{aligned} \tag{4.22}$$

where $k_i^\vee = (\boldsymbol{\alpha}_i^2/2) k_i$, which shows that $\tau_0^{(0)}(t), \ldots, \tau_r^{(0)}(t)$ are the Hirota tau-functions in this case.

In general, when $\widehat{s} \subset \widehat{g}_{\geq 0}(\mathbf{s})$, as in (4.5), the construction presented in sections 2 and 3 can still be generalized by introducing an auxiliary gradation $\mathbf{s}^* \preceq \mathbf{s}$ with respect to the partial ordering of [15]. Then, the new Lax operators would be defined such that

$$A_N \in \widehat{g}_{\geq 0}(\mathbf{s}^*) \cap \widehat{g}_{\leq N}(\mathbf{s}) , \tag{4.23}$$

and the analogue of the dressing transformation (2.16) involves the factorization

$$\Psi\, h\, \Psi^{-1} \;\in\; \widehat{G}_{-}(\mathbf{s}^*)\; \widehat{G}_{0}(\mathbf{s}^*)\; \widehat{G}_{+}(\mathbf{s}^*)\,. \tag{4.24}$$

Now, if $\widehat{s}$ is of the form given by eq. (4.5), the orbit of the vacuum solution (4.6) provides solutions for the generalized (partially modified) KdV hierarchies of [15], and it leads to the corresponding generalizations of the Hirota tau-functions [16].

In addition, the form of the new Lax operator (4.23), is invariant under the gauge transformations [15]

$$\mathcal{L}_N \;\mapsto\; U\mathcal{L}_N U^{-1} = U\left(\frac{\partial}{\partial t_N} - A_N\right)U^{-1} \tag{4.25}$$

where U is an exponentiation of elements of the algebra

$$P \equiv \widehat{g}_{0}(\mathbf{s}^*) \cap \widehat{g}_{\leq 0}(\mathbf{s}) \tag{4.26}$$

In terms of the associated linear problem, the gauge transformation (4.25) corresponds to $\Psi \mapsto U\Psi$, with $U \in P$. This opens the possibility of considering different gauge equivalent definitions of the dressing transformation (2.16), (with the decomposition (2.15) being replaced by (4.24)), *e.g.*,

$$\begin{aligned} \Psi \;\mapsto\; U\Psi^h &= [(\Psi\, h\, \Psi^{-1})_{-} U^{-1}]^{-1}\, \Psi \\ &= [U(\Psi\, h\, \Psi^{-1})_0]\, (\Psi\, h\, \Psi^{-1})_{+}\, \Psi\, h^{-1}\,, \end{aligned} \tag{4.27}$$

leading to gauge equivalent solutions of the zero-curvature equations. It is worth pointing out that the group of gauge tranformations (4.25) is not trivial even if $\mathbf{s}^* = \mathbf{s}$. However, along the paper, we only consider the dressing transformations defined by (2.16), which is equivalent to a gauge fixing prescription for the transformations (4.25) (an alternative prescrition is used, *e.g.*, in [27]).

4.2 Generalized non-abelian Toda equations.

Now, for a given Heisenberg subalgebra $\mathcal{H}[w]$ of $g^{(1)}$, let us choose a positive integer l and consider the vacuum solution

$$A_l^{(\mathrm{vac})} \;=\; \Lambda_l\,, \quad A_{-l}^{(\mathrm{vac})} \;=\; \Lambda_{-l} \,+\, l\, t_l\, c\,, \tag{4.28}$$

associated to the subalgebra generated by $\Lambda_{\pm l} \in \mathcal{H}[w] \cap \widehat{g}_{\pm l}(\mathbf{s}^w)$. In (2.10), this solution corresponds to $l_+ = l$, $l_- = 0$, $(-l)_+ = 0$, $(-l)_- = -l$, $c_l^j = \delta_{j,l}$, $c_{-l}^j = \delta_{j,-l}$, $\rho_l = 0$, and $\rho_{-l} = l\, t_l$, and it is equivalent to the following solution of the associated linear problem

$$\Psi^{(\mathrm{vac})} \;=\; \exp\Big(\Lambda_l\, t_l \,+\, \Lambda_{-l}\, t_{-l} \,+\, \frac{1}{2}\, l\, t_l\, t_{-l}\, c\Big)\,. \tag{4.29}$$

If $l > 1$, we will only be interested in the orbit of solutions generated by the dressing transformations associated with the elements of the subgroup $\widehat{G}^{(l)}$ formed by exponentiating the subalgebra

$$\widehat{g}^{(l)} = \bigoplus_{k\in\mathbb{Z}} \widehat{g}_{kl}(\mathbf{s}^w) \,. \tag{4.30}$$

However, let us indicate that the orbit generated by the full Kac-Moody group acting on (4.28) provides solutions for the generalized affine non-abelian Toda equations of [19]. Then, $\Psi^{(\mathrm{vac})}$, h, and, consequently, $\Psi^{(\mathrm{vac})}\, h\, \Psi^{(\mathrm{vac})\,-1}$ are in $\widehat{G}^{(l)}$, which implies

$$\Theta \;=\; \exp\biggl(\sum_{k\in\mathbb{Z}>0} \theta_{-kl}\biggr) \;=\; 1 + \theta_{-l} + \cdots \,\in\, \widehat{G}_{-}^{(l)}(\mathbf{s}^w)\,, \quad \theta_{-kl} \in \widehat{g}_{-kl}(\mathbf{s}^w)\,,$$

$$\Upsilon = \exp\biggl(\sum_{k\in\mathbb{Z}>0} \zeta_{kl}\biggr) \;=\; 1 + \zeta_{l} + \cdots \,\in\, \widehat{G}_{+}^{(l)}(\mathbf{s}^w)\,, \quad \zeta_{kl} \in \widehat{g}_{kl}(\mathbf{s}^w)\,. \tag{4.31}$$

Therefore, using eqs. (2.22), the orbit of solutions generated by the group of dressing transformations acting on (4.28) is given by

$$A_l^h \;=\; \Lambda_l \;+\; [\theta_{-l}, \Lambda_l] \tag{4.32}$$

$$\;=\; \Lambda_l \;-\; B^{-1}\partial_l B\,, \tag{4.33}$$

$$A_{-l}^h \;=\; \Lambda_{-l} \;+\; \partial_{-l}\theta_{-l} \;+\; l\, t_l\, c \tag{4.34}$$

$$\;=\; B^{-1}\Lambda_{-l}B \;+\; l\, t_l\;\, c\,. \tag{4.35}$$

Using eqs. (4.33) and (4.35), the zero-curvature equation $[\mathcal{L}_l^h, \mathcal{L}_{-l}^h] = 0$ becomes

$$\partial_l^-\bigl(B^{-1}\partial_l^+ B\bigr) \;=\; \bigl[\Lambda_l\,,\, B^{-1}\Lambda_{-l}B\bigr] - l\, c\,, \tag{4.36}$$

which shows that $\widehat{B} = e^{l\, t_l\, t_{-l}\, c}\, B$ is a solution of a generalized non-abelian affine Toda equation where $\pm t_{\pm l}$ play the role of the light-cone variables $x_\pm = x \pm t$, and $\widehat{B}$ is the Toda field [18,21,20,17,29]. It is well known that these equations can be understood as the classical equations-of-motion of certain two-dimensional relativistic field theories, and Lorentz invariance is manifested through the symmetry transformation $x_\pm \to \lambda^{\pm 1} x_\pm$.

However, the solutions provided by the dressing transformation method satify additional constraints. Actually, the comparison of (4.32) with (4.33) shows that

$$B^{-1}\, \partial_l\, B \;=\; [\Lambda_l\,,\, \theta_{-l}] \in \mathrm{Im}(\mathrm{ad}\,\Lambda_l)\,, \tag{4.37}$$

and (2.22) and (4.35) implies that

$$\partial_{-l}\, B\, B^{-1} \;=\; -\,[\Lambda_{-l}\,,\, \zeta_{-l}] \in \mathrm{Im}(\mathrm{ad}\,\Lambda_{-l})\,. \tag{4.38}$$

These constraints break the well known chiral symmetry

$$B \mapsto h_-(t_{-l})\, B\, h_+(t_l) \tag{4.39}$$

of the affine Toda equation (4.36), where $h_\pm(t_{\pm l})$ take values in the subgroups $\widehat{G}_\pm$ formed by exponentiating the subalgebras $\mathrm{Ker}(\mathrm{ad}\,\Lambda_{\pm l}) \cap \widehat{g}_0(\mathbf{s}^w)$. Moreover, eqs. (4.37) and (4.38) have a nice interpretation as gauge-fixing conditions for certain local (chiral) symmetries of the underlying two-dimensional field theory [17,30].

The dressing transformation method allows one to relate the resulting solutions of the non-abelian Toda (4.36) and generalized mKdV (4.11) equations. The crucial observation is that the Lax operator $\mathcal{L}_l^h$ corresponding to (4.33) can also be viewed as the Lax operator of a generalized mKdV hierarchy. Then, according to (4.8), the relation is $\Lambda = \Lambda_l$, $x = t_l \equiv x_+$, and $\widetilde{q} = -B^{-1}\partial_l B$. This provides a (locally) non-invertible map from solutions of the non-abelian Toda equation (B) into solutions of the mKdV equation ($\widetilde{q}$), which generalizes the known relation between solutions of the sine-Gordon and mKdV equations. However, if $l > 1$, notice that the resulting mKdV Lax operator is constrained by the condition $\widetilde{q} = -B^{-1}\partial_l B \in \widehat{g}_0(\mathbf{s}^w)$, a constraint that is compatible only with a subset of the flows that define the mKdV hierarchy. In order to make this relation concrete, let us consider the subalgebra

$$\widehat{s}^{\,\dagger} = \Big[\mathrm{Cent}\big(\mathrm{Ker}(\mathrm{ad}\,\Lambda_l)\big) \cap \bigoplus_{k\in\mathbb{Z}\geq 0} \widehat{g}_{kl}(\mathbf{s}^w)\Big] \cup \mathbb{C}\,\Lambda_{-l}\,, \tag{4.40}$$

and the associated vacuum solution

$$A_{kl}^{(\mathrm{vac})} = \begin{cases} \Lambda_{kl} & \text{if } k \geq 0\,, \\ \Lambda_{-l} + l\, t_l\, c & \text{if } k = -1\,. \end{cases} \tag{4.41}$$

Then, the orbit generated by the group of dressing transformations induced by the elements of $\widehat{G}^{(l)}$ provide joint solutions of both the non-abelian Toda equation (4.36) and the mKdV hierarchy of equations restricted to the flows generated by the times t_{kl}. [6] Recall that, both in the mKdV and non-abelian Toda equations, the solutions provided by the dressing transformation method satisfy additional constraints. In the generalized mKdV equations, these constraints are given by the condition that $h \in \widehat{g}_{<0}(\mathbf{s}^w)$ (see eq. (4.10) and the discussion leading to (4.13)). As for the non-abelian Toda equation, the so obtained solutions satisfy the identities (4.37) and (4.38). However, since, in this case, $\widetilde{q} = -B^{-1}\partial_l B \in \widehat{g}_0(\mathbf{s}^w)$, eq. (4.10) implies that $h \in \widehat{g}_{\leq 0}(\mathbf{s}^w)$, and that

[6] A similar construction can be used to connect the solutions of the full generalized mkdV hierarchies with the solutions of the generalized non-abelian Toda equations of [19].

$\widetilde{q} = \mathrm{P}_{0[\mathbf{s}^w]}(h) + [\Lambda_l, \varphi_{-l}]$, where $\Phi = 1 + \varphi_{-l} + \cdots$. All this shows that the constraint $h \in \widehat{g}_{<0}(\mathbf{s}^w)$ corresponds precisely to eq. (4.37). In contrast, eq. (4.38) involves the time t_{-l} and, hence, it does not have an interpretation in terms of the mKdV hierarchy associated with $\widetilde{q}$.

Finally, let us discuss the tau-functions for these non-abelian Toda hierarchies. Eqs. (4.33) and (4.35) show that the non-abelian Toda equation (4.36) is a partial differential equation for the Toda field B. Actually, this remains true if we consider the hierarchy of equations associated with the vacuum solution (4.41). Therefore, in this case, the generalized Hirota tau-functions correspond to the set of variables $\tau_{\mu_0,\mu_0'}(t)$, and the change of variables is provided by eq. (3.2). This shows that the vectors $|\mu_0\rangle$ have to be chosen in some integrable representation of $g^{(1)}$ such that they form a faithful representation of $\widehat{G}_0(\mathbf{s}^w)$. It is worth mentioning that eqs. (4.32) and (4.34) suggest the possibility of describing these hierarchies in terms of the components of Θ. Actually, those equations manifest the existing relations between the tau-functions $\tau_{\mu_0,\mu_0'}(t)$ and the components of $|\tau_i^R(t)\rangle$ that provide the tau-functions of the associated mKdV hierarchy. However, these relations are non-local and, therefore, not very useful in practice in the general case.

Eq. (3.1) shows that the proposed generalized tau-functions $\tau_{\mu_0,\mu_0'}(t)$ are precisely the matrix elements involved in the solitonic specialization of the Leznov-Saveliev solution of [26]. Therefore, using the map between the solutions of non-abelian Toda and mKdV equations, the dressing transformation method relates the resulting orbit of solutions of the latter with the solitonic specialization of the Leznov-Saveliev solution, originally formulated in the context of affine Toda equations. Then, since it can be justified that the solitonic specialization singles the soliton solutions out from the general Leznov-Saveliev solution [24,25,31], the observed relation supports the conjecture that the orbits of solutions generated by the group of dressing transformations actually contain all the multi-soliton solutions of the equations.

The simplest example is provided by the abelian affine Toda equations, which are related with the Drinfel'd-Sokolov mKdV hierarchies discussed in section 4.1. Therefore, they are recovered from the principal Heisenberg subalgebra and $l = 1$, with

$$\Lambda_1 = \sum_{i=0}^{r} e_i^+ \quad \text{and} \quad \Lambda_{-1} = \sum_{i=0}^{r} k_i^\vee \, e_i^- \, . \tag{4.42}$$

The Toda field takes values in $\widehat{g}_0(\mathbf{s}_p)$, which is generated by $h_0, \ldots, h_r$. This implies that

$$B = \exp\left(-\sum_{i=0}^{r} \phi_i \, h_i\right) = \exp\Big(-\boldsymbol{\phi} \cdot \boldsymbol{H}^{(0)} - \phi_0 \, c\Big) \, , \tag{4.43}$$

and, hence, the generalized Hirota equations are just $\tau_{v_i,v_i}(t) = \tau_i^{(0)}(t)$, for $i = 0, \ldots, r$. The relation between the components of B and the tau-functions follows from eq. (3.2)

$$\begin{aligned} \langle v_i | B^{-1} | v_i \rangle &= \mathrm{e}^{\phi_i} \\ &= \tau_i^{(0)}(t) \,, \quad \text{for} \quad i = 0, \ldots, r \,, \end{aligned} \tag{4.44}$$

which leads to

$$\boldsymbol{\phi} \;=\; \sum_{i=0}^{r} \ln \tau_i^{(0)}(t)\, \frac{2}{\boldsymbol{\alpha}_i^2}\, \boldsymbol{\alpha}_i \,, \quad \phi_0 \;=\; \ln\, \tau_0^{(0)}(t) \,, \tag{4.45}$$

which, compared with (4.22), exhibits the relation with the mKdV hierarchies, and agrees with the change of variables (1.2) used in [3,4,5,6].

5. Conclusions

In this paper, we have studied a special type of solutions of a large class of non-linear integrable zero-curvature equations. The class of integrable models is constructed from affine (both twisted and non-twisted) Kac-Moody algebras, and is characterized by exhibiting trivial solutions, referred to as "vacuum solutions", such that the corresponding Lax operators take values in some abelian subalgebra up to the central term. Then, we have considered the orbits of solutions generated by the group of dressing transformations acting on those vacua. It is important to remark that the relevant integrable models are not constructed explicitly. In contrast, their zero-curvature equations are found only as the equations satisfied by this particular class of solutions. This is similar to the tau-function approach of [12] where the equations defining the integrable hierarchy of bi-linear Hirota equations are derived from the property that their solutions lie in the orbit of a highest-weight vector generated by a Kac-Moody group. The resulting class of integrable models include the generalizations of the Drinfel'd-Sokolov hierarchies of mKdV (and KdV) type constructed in [15], and the generalizations of the sine-Gordon equation known as abelian and non-abelian affine Toda equations [18,17,19,20].

The motivation for studying this particular type of solutions is to find the generalizations of the Hirota tau-functions for the relevant integrable systems. First of all, it is generally assumed that the orbit of solutions generated by the group of dressing transformations contain all the multi-soliton solutions of the integrable hierarchy [27]. Then, according to the method of Hirota, the generalized tau-functions provide an alternative set of variables that largely simplify the task of constructing the multi-soliton solutions. In

this case, we have identified those new variables with specific matrix elements evaluated in the integrable highest-weight representations of the Kac-Moody algebra.

In particular, for the generalizations of the Drinfel'd-Sokolov hierarchies of mKdV (and KdV) type, our results constitute a generalization of the results of [16] to the general case when the relevant integrable representations are neither of level-one nor of vertex type. Moreover, for the non-abelian affine Toda equations, our results show that the suitable generalizations of the Hirota tau-functions correspond to the matrix elements involved in the solitonic specialization of the general Leznov-Saveliev solution [26]. Actually, this is a remarkable result since it links the orbits of solutions under consideration with the solitonic specialization of [26]. Then, since the solitonic specialization arises as a prescription to single the multi-soliton configurations out from the general solution, our result supports the conjecture that all the multi-soliton solutions lie in the orbit generated by the group of dressing transformation acting on some vacuum.

In this paper we have only considered integrable systems of zero curvature equations constructed from Kac-Moody algebras. However, there are many other important integrable hierarchies formulated by means of pseudo-differential operators, and it would be interesting to investigate the implications of our results for the definition of generalized Hirota tau-functions in those cases [32]. In particular, it has been recently shown in ref. [33] how matrix generalizations of both the Gelfand-Dickey and the constrained KP hierarchies can be recovered from the construction of [15]. Therefore, at least in these important cases, it should be possible to translate directly our definition of tau-functions into the context of those integrable systems.

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Appendix: Integrable Highest-Weight Representations

An affine Kac-Moody algebra $\mathfrak{g}$ of rank r is defined by a generalized Cartan matrix a of affine type of order $r+1$ (and rank r), and is generated by $\{h_i, e_i^{\pm}, i = 0, \ldots, r\}$ and d subject to the relations [34]

$$\begin{aligned} [h_i\,,\, h_j] &= 0\,, \quad [h_i\,,\, e_j^{\pm}] \;=\; \pm a_{i,j}\, e_j^{\pm}\,, \\ [e_i^{+}\,,\, e_j^{-}] \;&=\; \delta_{i,j}\, h_i\,, \quad (\mathrm{ad}\, e_i^{\pm})^{1-a_{i,j}}\, (e_j^{\pm}) \;=\; 0\,, \\ [d\,,\, h_i] &= 0\,, \quad [d\,,\, e_i^{\pm}] \;=\; \pm \delta_{i,0}\, e_0^{\pm}\,. \end{aligned} \tag{A.1}$$

The elements $e_i^{\pm}$ are Chevalley generators, and $\{h_0, \ldots, h_r, d\}$ span the Cartan subalgebra of $\mathfrak{g}$. The algebra $\mathfrak{g}$ has a centre $\mathbb{C}\, c$ generated by the central element $c = \sum_{i=0}^{r} k_i^{\vee}\, h_i$ where $k_i^{\vee}$ are the labels of the dual Dynkin diagram of $\mathfrak{g}$ (the dual Kac labels), and in all cases $k_0^{\vee} = 1$.

The different $\mathbb{Z}$-gradations of $\mathfrak{g}$ are labelled by sets $\mathbf{s} = (s_0, \ldots, s_r)$ of non-negative integers. Then the gradation is induced by a derivation $d_{\mathbf{s}}$ such that

$$[d_{\mathbf{s}}\,,\, h_i] \;=\; [d_{\mathbf{s}}\,,\, d] \;=\; 0\,, \quad [d_{\mathbf{s}}\,,\, e_i^{\pm}] \;=\; \pm s_i e_i^{\pm}\,. \tag{A.2}$$

In particular, the derivation d corresponds to the, so called, homogeneous gradation $\mathbf{s} = (1, 0, \ldots, 0)$.

The definition of integrable representations makes use of the following property. An element $x \in \mathfrak{g}$ is said to be "locally nilpotent" on a given representation if for any vector $|v\rangle$ there exists a positive integer N_v such that $x^{N_v}|v\rangle = 0$. Then, an integrable highest-weight representation $L(\mathbf{s})$ of $\mathfrak{g}$ is a highest-weight representation of $\mathfrak{g}$ where the Chevalley generators are locally nilpotent [34]. It can be proven that $L(\mathbf{s})$ is irreducible and that $|v_{\mathbf{s}}\rangle$ is the unique highest-weight vector of $L(\mathbf{s})$.

The highest-weight vector of $L(\mathbf{s})$ can be labelled by a gradation, $\mathbf{s} = (s_0, s_1, \ldots, s_r)$ such that

$$e_i^{+}\, |v_{\mathbf{s}}\rangle \;=\; \left(e_i^{-}\right)^{s_i+1}\, |v_{\mathbf{s}}\rangle \;=\; 0\,, \tag{A.3}$$

$$h_i\, |v_{\mathbf{s}}\rangle \;=\; s_i\, |v_{\mathbf{s}}\rangle\,, \qquad d_{\mathbf{s}}\, |v_{\mathbf{s}}\rangle = 0\,, \tag{A.4}$$

for all $i = 0, \ldots, r$. Notice that the eigenvalue of $d_{\mathbf{s}}$ is arbitrary, and that $d_{\mathbf{s}}$ can be diagonalized acting on $L(\mathbf{s})$. The eigenvalue of the centre c on the representation $L(\mathbf{s})$ is

known as the level k

$$c\,|v_{\mathbf{s}}\rangle \;=\; \sum_{i=0}^{r} k_i^{\vee}\, h_i\, |v_{\mathbf{s}}\rangle \;=\; \left(\sum_{i=0}^{r} k_i^{\vee}\, s_i\right) |v_{\mathbf{s}}\rangle\;, \tag{A.5}$$

hence $k = \sum_{i=0}^{r} k_i^{\vee}\, s_i \in \mathbb{Z} \geq 0$. On $L(\mathbf{s})$ there is a notion of orthogonality by means of a (unique) positive definite Hermitian form H such that $H(|v_{\mathbf{s}}\rangle, |v_{\mathbf{s}}\rangle) \equiv \langle v_{\mathbf{s}}|v_{\mathbf{s}}\rangle = 1$. Finally, $L(\mathbf{s})$ can be "integrated" to a representation of the Kac-Moody group $\widehat{G}$, which is then generated just by the exponentials of the generators of $\mathfrak{g}$ [35,34,28].

We will use the notation $|v_i\rangle$ for the highest-weigh vector of the "fundamental" representation $L(i)$ where $s_j = \delta_{j,i}$, and d_i for the corresponding derivation. In terms of these fundamental highest-weight vectors, $|v_{\mathbf{s}}\rangle$ can be decomposed as

$$|v_{\mathbf{s}}\rangle \;=\; \bigotimes_{i=0}^{r} \left\{ |v_i\rangle^{\otimes s_i} \right\}. \tag{A.6}$$

It follows from its definition that the highest-weight vector of $L(\mathbf{s})$ is annihilated by $\widehat{g}_{\,>0}(\mathbf{s})$, and that it is an eigenvector of $\widehat{g}_{\,0}(\mathbf{s})$ with eigenvalues

$$\begin{aligned} h_i\,|v_{\mathbf{s}}\rangle \;&=\; s_i\,|v_{\mathbf{s}}\rangle\;, \qquad d_{\mathbf{s}}\,|v_{\mathbf{s}}\rangle \;=\; 0\;, \\ e_j^{-}\,|v_{\mathbf{s}}\rangle \;&=\; 0 \quad \text{when} \quad s_j = 0\,. \end{aligned} \tag{A.7}$$

Then, the representation of the subgroups $\widehat{G}_{\,+}(\mathbf{s})$, $\widehat{G}_{\,-}(\mathbf{s})$, and $\widehat{G}_{\,0}(\mathbf{s})$ on $L(\mathbf{s})$ are actually generated by exponentiating the generators of $\widehat{g}_{\,>0}(\mathbf{s})$, $\widehat{g}_{\,<0}(\mathbf{s})$, and $\widehat{g}_{\,0}(\mathbf{s})$, respectively.

For simply laced affine Kac-Moody algebras, all the fundamental integrable representations of level-one are isomorphic to the basic representation $L(0)$, which can be realized in terms of vertex operators acting on Fock spaces [28,36]. Then, the other fundamental integrable representations of level > 1 can be realized as submodules in the tensor product of several fundamental level-one representations. Moreover, the fundamental integrable representations of non-simply laced Kac-Moody algebras can be constructed from those of the simply laced algebras by folding them [25,37].

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